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Ilaria Di Dedda

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First maps we want to study: Lefschetz fibrations

def A map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a Lefschetz fibration if:

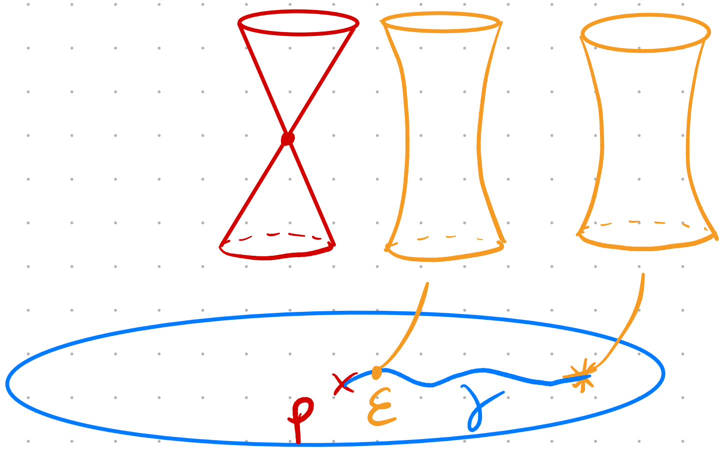
- it has finitely many isolated non-degenerate singularities
  $$\{p_1, \dots, p_k\} \subset \mathbb{C}^2$$
- near each p_i, there are charts on which
  $$f(x,y) = x^2 + y^2$$
- away from $\{p_1, \dots, p_k\}$, f is a locally trivial fibre bundle

$f^{-1}(*) =$ Riemann surface

example:

$f(x,y) = x^2 + y^2$

$$\begin{array}{c} \mathbb{C}^2 \\ \downarrow \\ \mathbb{C} \end{array}$$



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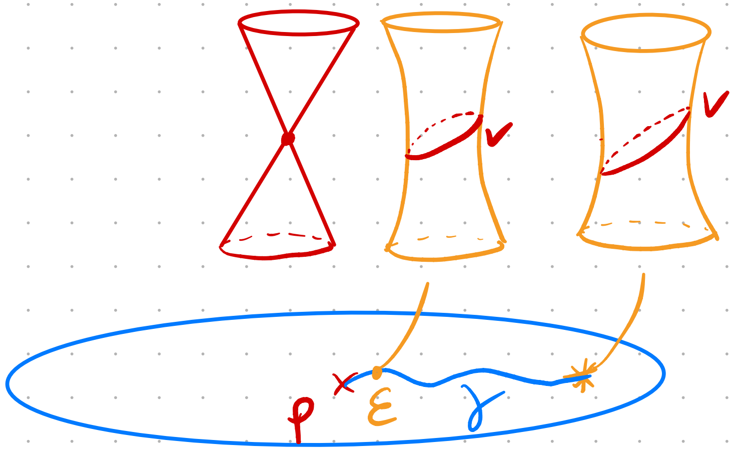
$$\mathbb{C}^2 \xrightarrow{\quad} \mathbb{C}$$



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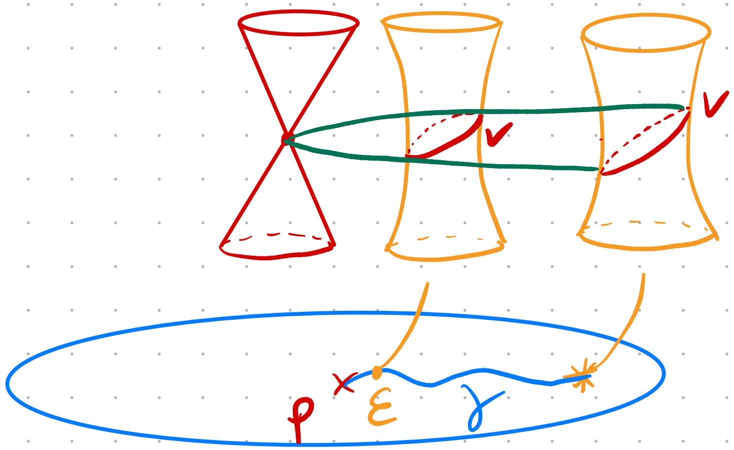

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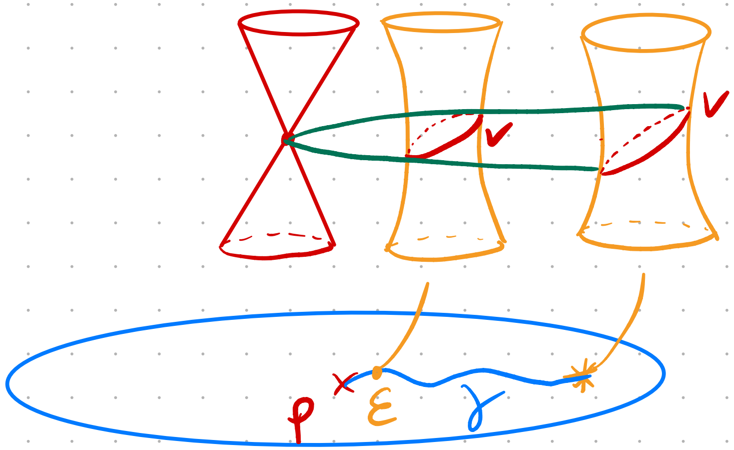

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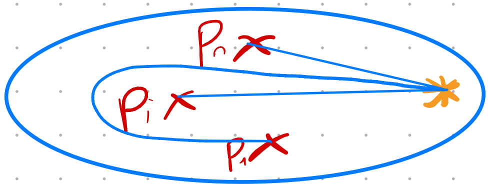

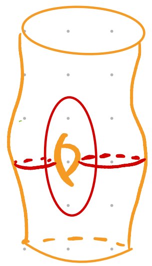

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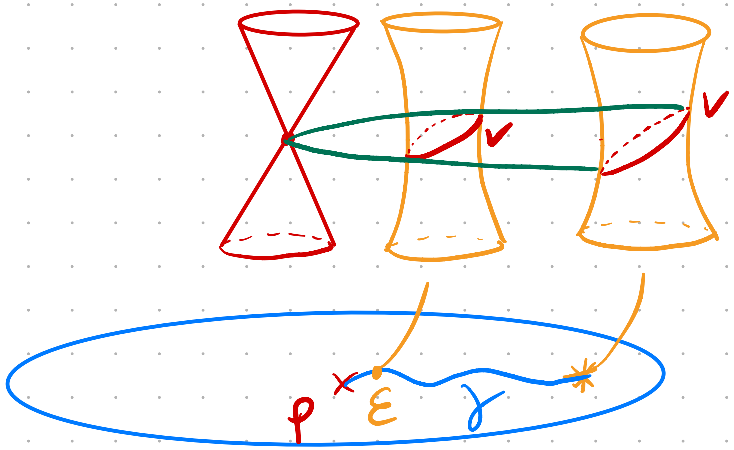

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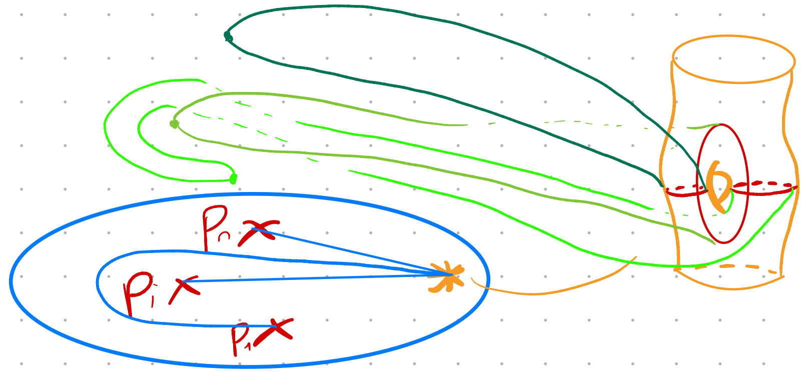

Facts:

- $\{D_i\}$ are ordered clockwise
- for $D_i < D_j$, define

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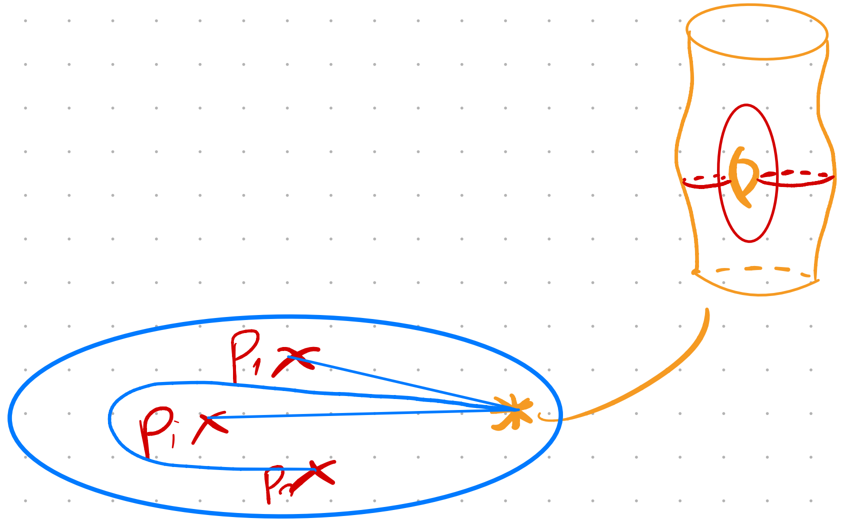

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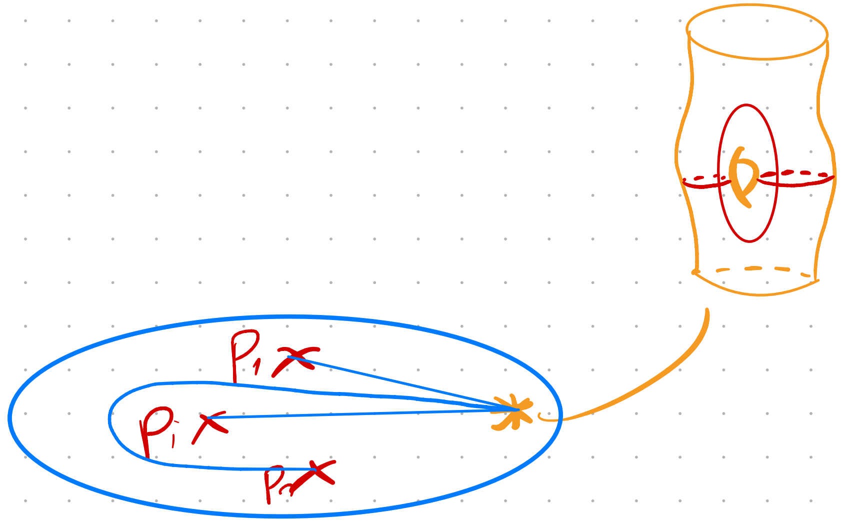

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Fact: the derived $\mathcal{F}(f)$ does not depend on the choices

Next step : Isolated singularities

def (In this talk) A curve singularity (CS) is an isolated polynomial curve singularity $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ (Crit $f = \{(0,0)\}$)
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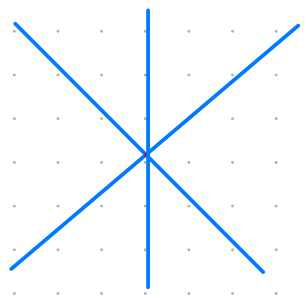

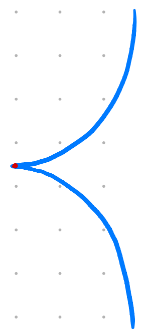

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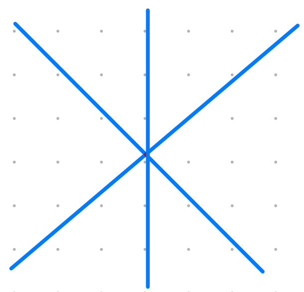

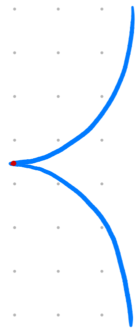

f CS $\longleftrightarrow$ $\{f(x,y) = 0\} \subset \mathbb{R}^2$

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A : use "good real morsifications" (A'Campo, Gusein-Zade
+ Keating)

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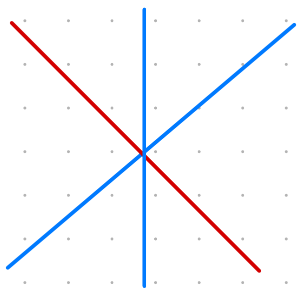

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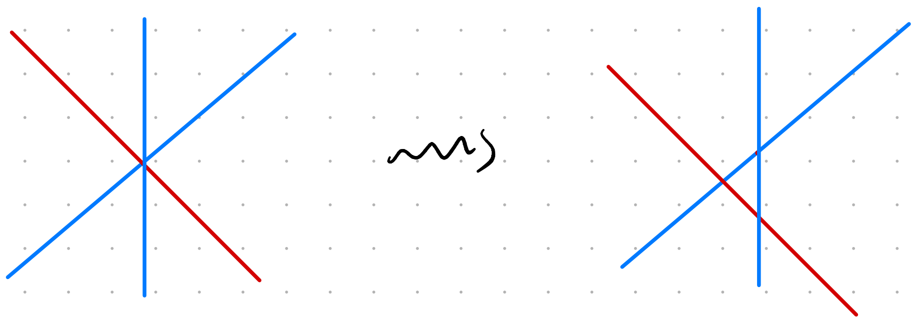

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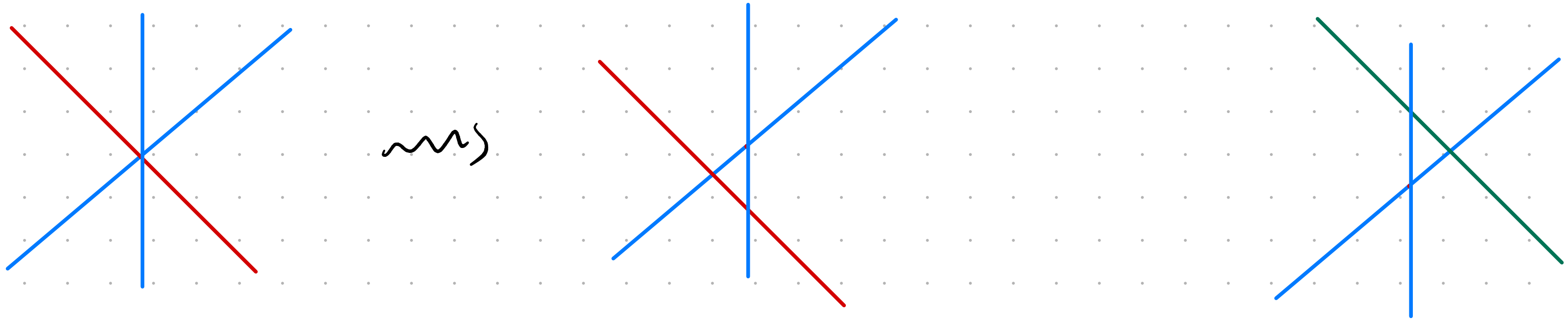
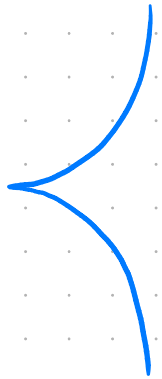


more examples :

- $x^5 - y^2$



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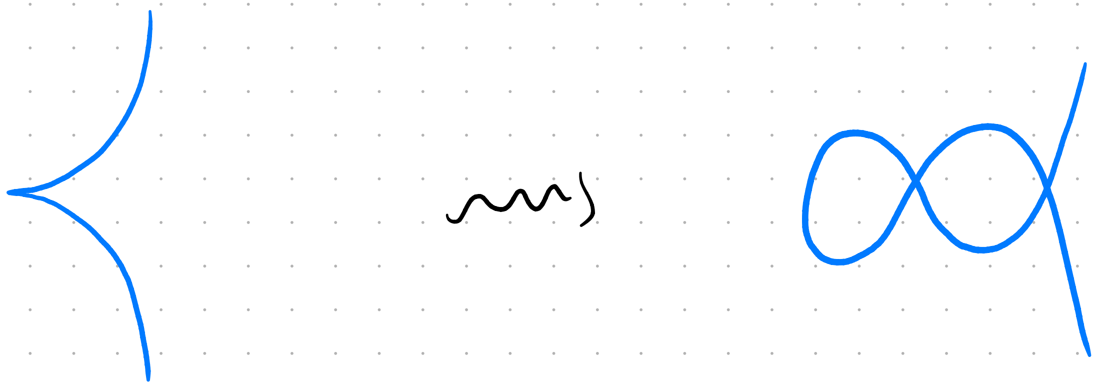
- $x^5 - y^2 \rightsquigarrow x(x+1)^2(x+2)^2 - y^2$



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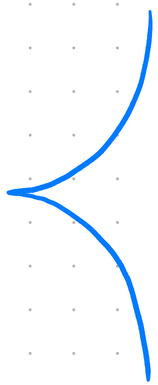
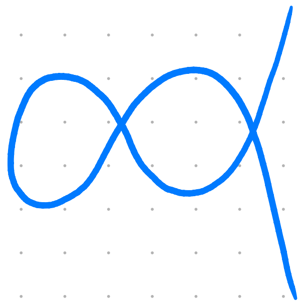
 $\rightsquigarrow$ 

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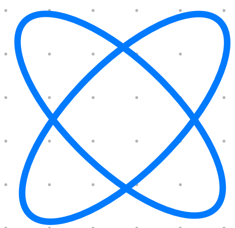
 $\rightsquigarrow$ 

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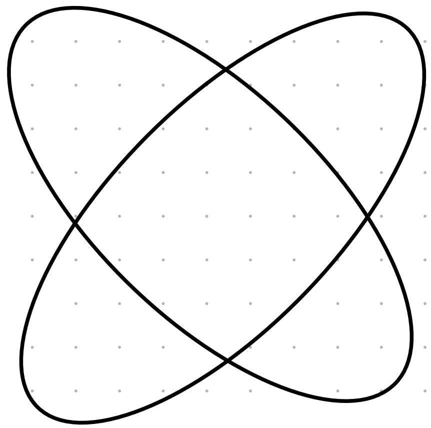

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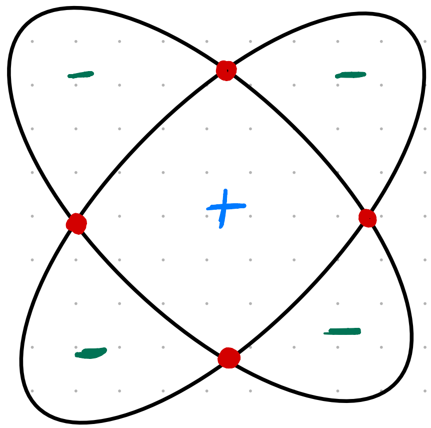

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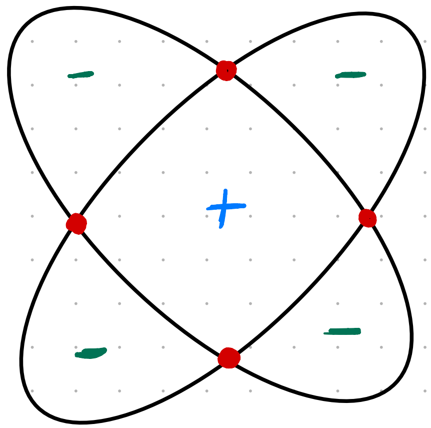

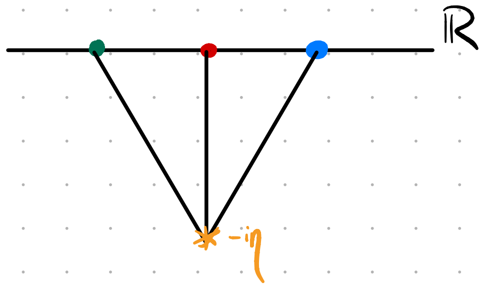

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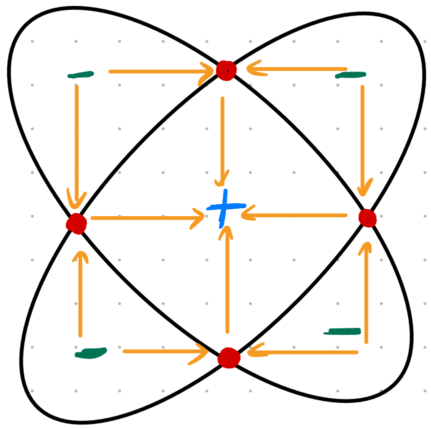

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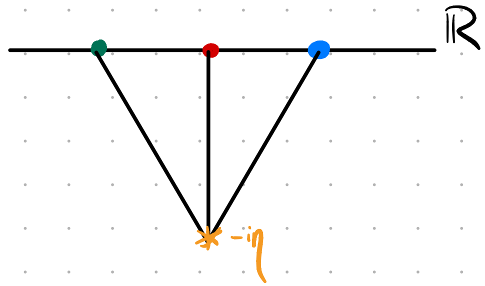

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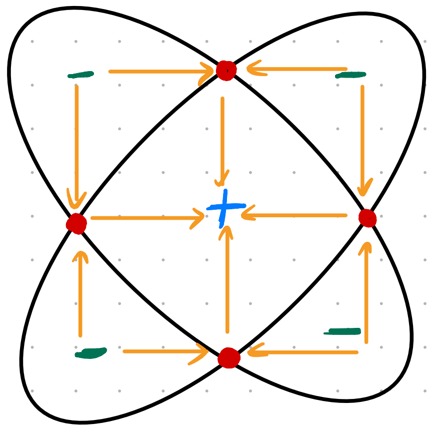

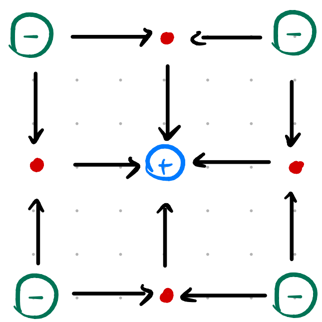

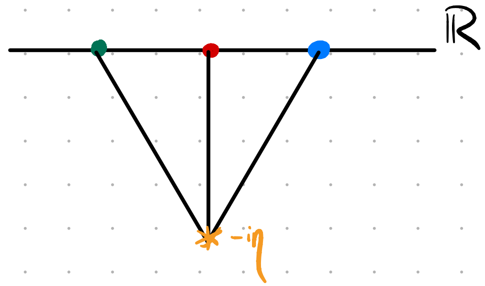

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Thank you!

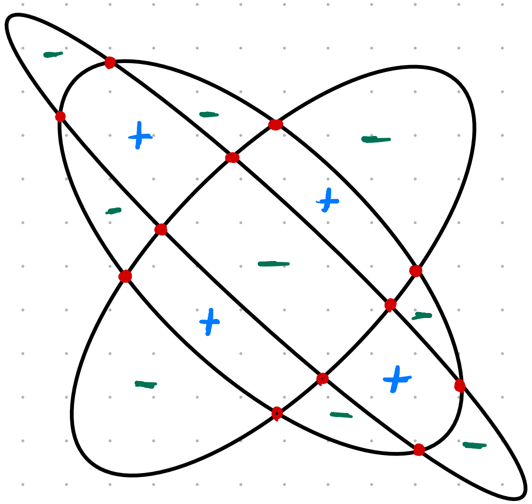
time permitting: $f_n : \mathbb{C}^2 \longrightarrow \mathbb{C}$, $S_2 \curvearrowright \mathbb{C}^2$, $\mathbb{C}^2/S_2 =: \mathrm{Sym}^2\mathbb{C} \simeq \mathbb{C}^2$
$\qquad (x,y) \longmapsto x^n+y^n$

$$x^n + y^n = \prod_{i=1}^{n/2} \left(x^2+y^2-\alpha_i\, xy\right)$$

$$\rightsquigarrow$$

$$\prod_{i=1}^{n/2} \left(x^2+y^2-\alpha_i\, xy - h_i\right)$$

$f_n : \mathbb{C}^2 \to \mathbb{C}$



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